COMBINATORICA
Akadémiai Kiadó — Springer-Verlag

COMBINATORICA 9 (2) 1989 153—160

AN EXTREMAL PROBLEM FOR GRAHAM—ROTHSCHILD PARAMETER WORDS

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Received August 1, 1986

This paper exposes connections between the theory of Möbius functions and extremal problems, extending ideas of Frankl and Pach [8]. Extremal results concerning the trace of objects in geometric lattices and Graham—Rothschild parameter posets are proved, covering previous results due to Sauer [16] and Perles and Shelah [17].

0. Introduction

Answering a question of Erdős, Perles, Shelah and Sauer proved the following

Theorem [16, 17]. *Let X be a finite set with $|X|=n$. Further let $\mathcal{F}\subseteq\mathcal{P}(X)$ be a subset of the powerset $\mathcal{P}(X)$ of X with $|\mathcal{F}|>\sum_{i=0}^{t}\binom{n}{i}$ for some nonnegative integer $t<n$. Then there exists a subset $T\subseteq X$ with $|T|=t+1$, such that for every subset $T_0\subseteq T$ there exists a set $F\in\mathcal{F}$ with $F\cap T=T_0$.*

In [8] this theorem arises as a corollary from theorems concerning Steiner-systems. For related results compare e.g. [13, 2, 3, 7].

In this paper we indicate some connections between the theory of Möbius functions (cf. [1]) and extremal problems. This leads to generalizations of Sauer's result for geometric lattices and Graham—Rothschild parameter words.

1. Null t-designs

Let $(X, \wedge, \vee)$ be a ranked finite lattice with minimal element $\mathbf{0}$ and maximal element $\mathbf{1}$. The underlying partial order is denoted by $\leqq$. For nonnegative integers l let $\binom{X}{l}=\{x\in X|rg(x)=l\}$ be the l'th level of X. The vector space of all real valued functions $f\colon X\to\mathbf{R}$ is denoted by $V(X)$. For $z\in X$ let $N(z)=|\{x\in[\mathbf{0},z]|\ \mu(x,z)\neq 0\}|$, where μ is the Möbius function of X.

Let t be a nonnegative integer. A function $f\colon X\to\mathbf{R}$ is a *null t-design* iff for every $x\in X$ with $rg(x)\leqq t$ it is valid $\sum_{z\in[x,\mathbf{1}]}f(z)=0$. A function $f\colon X\to\mathbf{R}$ is a

AMS subject classification (1980): 05 C 35.

maximal null t-design iff f is a null t-design but not a null $(t+1)$-design. Clearly, null t-designs form subspaces of $V(X)$. For the powerset lattice $\mathscr{P}(n)$ of an n element set these were studied e.g. in [12], [10], [5], [6], [8] and [9].

Theorem 1. *Let $f\colon X\to\mathbf{R}$ be a maximal null t-design with $t<rg(\mathbf{1})$. Then*

$$\left|\{x\in X\,|\,f(x)\neq 0\}\right| \geqq \min_{z\in\binom{X}{t+1}} N(z)$$

and this bound is sharp.

Proof of Theorem 1. We show first that equality can be attained. For $x\in X$ let $\chi_x\colon X\to\{0,1\}$ denote the indicator function w.r.t. x defined by $\chi_x(y)=1$ iff $x=y$. Take $z\in\binom{X}{t+1}$ such that $N(z)=\min_{x\in\binom{X}{t+1}} N(x)$. Consider the function $f\colon X\to\mathbf{R}$ with $f=\sum_{x\in[0,z]}\mu(x,z)\cdot\chi_x$. Clearly, $\left|\{x\in X\,|\,f(x)\neq 0\}\right|=N(z)$. We prove that f is a maximal null t-design. Let $y\in X$ with $rg(y)\leqq t$. By definition of f and μ we conclude:

$$\begin{aligned}\sum_{v\in[y,1]} f(v) &= \sum_{v\in[y,1]}\sum_{x\in[0,z]}\mu(x,z)\cdot\chi_x(v)\\ &= \sum_{v\in[y,z]}\mu(v,z)\\ &= 0\end{aligned}$$

while on the other hand we have

$$\begin{aligned}\sum_{v\in[z,1]} f(v) &= \sum_{v\in[z,1]}\sum_{x\in[0,z]}\mu(x,z)\cdot\chi_x(v)\\ &= \mu(z,z)\\ &= 1.\end{aligned}$$

Thus f is a maximal null t-design.

Now we prove the desired inequality. For a function $f\colon X\to\mathbf{R}$ and $x\in X$ let $f_x\colon[0,x]\to\mathbf{R}$, the trace of f w.r.t. x, be defined by

$$f_x(v)=\sum_{\substack{x\wedge w=v\\ w\in X}} f(w)$$

for $v\in[0,x]$.

Fact. *Let $f\colon X\to\mathbf{R}$ be a null t-design and let $x\in X$. Then f_x is a null t-design on $[0,x]$.*

Proof of Fact. Let $u\in[0,x]$ with $rg(u)\leqq t$. By the definition of f_x and the assumption, that f is a null t-design it follows:

$$\begin{aligned}\sum_{v\in[u,x]} f_x(v) &= \sum_{v\in[u,x]}\sum_{\substack{x\wedge w=v\\ w\in X}} f(w)\\ &= \sum_{w\in[u,1]} f(w)\\ &= 0. \quad \blacksquare\end{aligned}$$

Let $f\colon X\to\mathbf{R}$ be a maximal null t-design with $t<rg(\mathbf{1})$. Then there exists $u\in\binom{X}{t+1}$ with $\sum_{x\in[u,1]} f(x)=c\neq 0$. By induction on $rg(u)-rg(r)$ we prove that for every $r\in[0,u]$ it is valid

$$f_u(r)=\mu(r,u)\cdot c.$$

For $r=u$ we have

$$f_u(u)=\sum_{\substack{w\wedge u=u\\ w\in X}} f(w)=c.$$

Suppose that for some $r\in[0,u)$ the statement is valid for all $s\in(r,u]$. By the fact f_u is a null t-design and since $rg(r)\leqq t$ we get by the inductive assumption:

$$\begin{aligned}0&=\sum_{s\in[r,u]} f_u(s)\\ &=f_u(r)+\sum_{s\in(r,u]} f_u(s)\\ &=f_u(r)+c\cdot\sum_{s\in(r,u]}\mu(s,u).\end{aligned}$$

Now $\mu(r,u)=-\sum_{s\in(r,u]}\mu(s,u)$ yields $f_u(r)=c\cdot\mu(r,u)$. Since $f_u(r)\neq 0$ implies that $f(x)\neq 0$ for some $x\in X$ with $x\wedge u=r$ we get

$$\left|\{x\in X\,|\,f(x)\neq 0\}\right|\geqq N(u).\quad\blacksquare$$

Denote by

— $\mathscr{P}(n)$ the powerset lattice of an n element set

— $\mathscr{L}(n,q)$ the lattice of linear subspaces of an n dimensional linear space over $GF(q)$

— $\mathscr{A}(n,q)$ the lattice of affine subspaces of an n dimensional vector space over $GF(q)$

— $\Pi(n)$ the lattice of partitions of an n element set.

— Let $\binom{n}{i}$, $\binom{n}{i}_q$, $q^{n-i+1}\binom{n}{i-1}_q$ and $S_{n,i}$ be the corresponding Whitney-numbers. Recall that $G_{n,q}=\sum_{i=0}^{n}\binom{n}{i}_q$ resp. $B_n=\sum_{i=0}^{n} S_{n,i}$ are the Galoisnumbers resp. Bellnumbers.

In [15] it has been shown that for finite geometric lattices X it is valid: $\mu(x,y)\neq 0$ for all $x,y\in X$ with $x\leqq y$. Applications of Theorem 1 to the above mentioned structures yield the following corollaries:

Corollary [8]. *Let $f\colon \mathscr{P}(n)\to\mathbf{R}$ be a nontrivial null t-design. Then*

$$\left|\{S\in\mathscr{P}(n)|f(S)\neq 0\}\right|\geqq 2^{t+1}.\quad\blacksquare$$

Corollary. *Let $f\colon \mathscr{L}(n,q)\to\mathbf{R}$ be a nontrivial null t-design. Then*

$$\left|\{U\in\mathscr{L}(n,q)|f(U)\neq 0\}\right|\geqq G_{t+1,q}.\quad\blacksquare$$

Corollary. *Let $f\colon \mathscr{A}(n,q)\to\mathbf{R}$ be a nontrivial null t-design. Then*

$$\left|\{U\in\mathscr{A}(n,q)|f(U)\neq 0\}\right|\geqq 1+\sum_{i=0}^{t} q^{t-i}\binom{t}{i}_q.\quad\blacksquare$$

Corollary. *Let f: $\Pi(n)\to\mathbf{R}$ be a nontrivial null t-design. Then*

$$\left|\{\pi\in\Pi(n)|f(\pi)\neq 0\}\right| \geqq B_{t+1}. \quad \blacksquare$$

Now we consider again arbitrary ranked finite lattices X.

Theorem 2. *Suppose that for all $x, y\in X$ with $x\leqq y$ it is valid $\mu(x, y)\neq 0$. Let $\mathcal{G}\subseteq X$ with $|\mathcal{G}|>\sum_{i=0}^{t}\left|\binom{X}{i}\right|$ for some $t<rg(\mathbf{1})$. Then there exists $y\in\binom{X}{t+1}$ such that for every $x\in[\mathbf{0}, y]$ there exists $g\in\mathcal{G}$ with $g\wedge y=x$.*

The family $\mathcal{G}=\bigcup_{i=0}^{t}\binom{X}{i}$ shows that this bound is sharp. The assumption $\mu(x, y)\neq 0$ for all $x, y\in X$ with $x\leqq y$ cannot be omitted as the following example indicates.

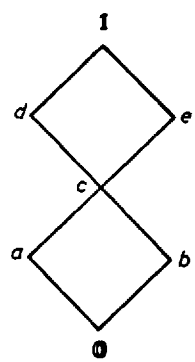

Fig. 1

For the lattice X indicated in the figure we have $\mu(\mathbf{0}, \mathbf{0})=\mu(\mathbf{0}, c)=1$, $\mu(\mathbf{0}, a)=$ $=\mu(\mathbf{0}, b)=-1$ and $\mu(\mathbf{0}, d)=\mu(\mathbf{0}, e)=\mu(\mathbf{0}, \mathbf{1})=0$. Let $\mathcal{G}=\{\mathbf{0}, a, d, e, \mathbf{1}\}$. There is no $g\in\mathcal{G}$ with $g\wedge c=b$.

Proof of Theorem 2. For $g\in\mathcal{G}$ let $f_g\in V(X)$ be a function defined by

$$f_g(x)=\begin{cases}1 & \text{if } x\leqq g \text{ and } rg(x)\leqq t\\ 0 & \text{else.}\end{cases}$$

Since the subspace of $V(X)$ generated by $\{f_g|g\in\mathcal{G}\}$ has dimension at most $\sum_{i=0}^{t}\left|\binom{X}{i}\right|$, there are reals $\alpha(g)$, $g\in\mathcal{G}$, not all zero such that $\sum_{g\in\mathcal{G}}\alpha(g)f_g=0$. Consider the function $h\in V(X)$ defined by

$$h(x)=\begin{cases}\alpha(x) & \text{if } x\in\mathcal{G}\\ 0 & \text{else.}\end{cases}$$

For $x\in X$ with $rg(x)\leqq t$ it is valid

$$\sum_{v\in[x,1]} h(v) = \sum_{\substack{g\in[x,1]\\ g\in\mathcal{G}}} \alpha(g) = \sum_{\substack{g\in[x,1]\\ g\in\mathcal{G}}} \alpha(g)\cdot f_g(x) =$$

$$= \sum_{\substack{g\in[x,1]\\ g\in\mathcal{G}}} \alpha(g) f_g(x) + \sum_{\substack{g\notin[x,1]\\ g\in\mathcal{G}}} \alpha(g) f_g(x) = \sum_{g\in\mathcal{G}} \alpha(g) f_g(x) = 0.$$

Thus $h\colon X\to\mathbf{R}$ is a nontrivial null t-design. As in the proof of Theorem 1 we find $y\in\binom{X}{t_0}$ with $t_0\geqq t+1$ such that $h_y(v)\neq 0$ for all $v\in[0,y]$ implying that for each $x\in\binom{X}{t+1}$ with $x\leqq y$ it is valid: for every $v\in[0,x]$ there exists $g\in\mathcal{G}$ with $g\wedge x=v$.

Corollary. *Let X be a ranked, finite geometric lattice. Let* $\mathcal{G}\subseteq X$ *with* $|\mathcal{G}|>\sum_{i=0}^{t}\left|\binom{X}{i}\right|$ *for some* $t<rg(\mathbf{1})$. *Then there exists* $y\in\binom{X}{t+1}$ *such that for every* $x\in[0,y]$ *there exists* $g\in\mathcal{G}$ *with* $g\wedge y=x$. ▌

For lattices X and Y let $X\cong Y$ denote that X and Y are isomorphic.

Corollary [16, 17]. *Let* $\mathcal{G}\subseteq\mathcal{P}(n)$ *be a family of subsets with* $|\mathcal{G}|>\sum_{i=0}^{t}\binom{n}{i}$ *for some* $t<n$. *Then there exists a* $(t+1)$*-element subset* $H\in\mathcal{P}(n)$ *such that* $\{H\cap G|G\in\mathcal{G}\}\cong\cong\mathcal{P}(t+1)$. ▌

Corollary. *Let* $\mathcal{G}\subseteq\mathcal{L}(n,q)$ *be a family of linear subspaces with* $|\mathcal{G}|>\sum_{i=0}^{t}\binom{n}{i}_q$ *for some* $t<n$. *Then there exists a* $(t+1)$*-dimensional linear subspace* $U\in\mathcal{L}(n,q)$ *such that* $\{U\cap G|G\in\mathcal{G}\}\cong\mathcal{L}(t+1,q)$. ▌

Corollary. *Let* $\mathcal{G}\subseteq\mathcal{A}(n,q)$ *be a family of affine subspaces with* $|\mathcal{G}|>1+\sum_{i=0}^{t}q^{n-i}\binom{n}{i}_q$ *for some* $t<n$. *Then there exists a t-dimensional affine subspace* $U\in\mathcal{A}(n,q)$ *such that* $\{U\cap G|G\in\mathcal{G}\}\cong\mathcal{A}(t+1,q)$. ▌

Corollary. *Let* $G\subseteq\Pi(n)$ *be a family of partitions with* $|\mathcal{G}|>\sum_{i=0}^{t}S_{n,i}$ *for some* $t<n$. *Then there exists a partition* $\pi\in\Pi(n)$ *having* $(n-t-1)$ *many blocks such that* $\{\pi\wedge\tau|\tau\in\mathcal{G}\}\cong\Pi(t+1)$. ▌

2. Graham—Rothschild Parameter Words

The concept of parameter words was introduced by Graham and Rothschild [11]. This combinatorial structure turned out to be a very fruitful tool in Ramsey Theory (compare e.g. [14]).

Let A be a finite alphabet. For nonnegative integers $m\leqq n$ and symbols $\lambda_0,\dots,\lambda_{m-1}$, serving as parameters with $A\cap\{\lambda_0,\dots,\lambda_{m-1}\}=\emptyset$, let $[A]\binom{m}{n}$ be the set of all mappings $f\colon\{0,\dots,n-1\}\to A\cup\{\lambda_0,\dots,\lambda_{m-1}\}$, which satisfy:

(i) $f^{-1}(\lambda_j)\neq\emptyset$ for every $0\leqq j<m$ and

(ii) $\min f^{-1}(\lambda_i)<\min f^{-1}(\lambda_j)$ for all $0\leqq i<j<m$.

Condition (i) means that all parameters $\lambda_0, \ldots, \lambda_{m-1}$ occur in the image of f and (ii) yields a rigid representation, i.e. the first occurrences of different parameters are in increasing order. Mappings $f \in [A]\binom{m}{n}$ are called *m-parameter words of length n over alphabet A*.

For example, a mapping $f \in [A]\binom{n}{0}$ describes just a point in A^n. The number of m-parameter words $f \in [A]\binom{n}{m}$ with $|A| = a$ is counted by the noncentral Stirling numbers $S_m^n(a)$ of the second kind, where

$$S_m^n(a) = \frac{1}{2\pi i} \oint \frac{x^n}{\prod_{i=0}^{m} (x-a-i)} \, dx;$$

these satisfy the Pascal identy

$$S_{m+1}^{n+1}(a) = S_m^n(a) + (a+m+1) \cdot S_{m+1}^n(a),$$

compare e.g. [4].

For parameter words $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ a composition $f \cdot g \in [A]\binom{n}{k}$ is defined by

$$f \cdot g(i) = \begin{cases} f(i) & \text{if } f(i) \in A \\ g(j) & \text{if } f(i) = \lambda_j. \end{cases}$$

This yields a partial ordering $\leqq$ on $\bigcup_{m=0}^{n} [A]\binom{n}{m}$. Let $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{n}{k}$. Then $f \geqq g$ iff there exists $h \in [A]\binom{m}{k}$ such that $f \cdot h = g$.

We illustrate this combinatorial structure for some special alphabets.

A = ∅:

Parameter words $f \in [\emptyset]\binom{n}{m}$ represent equivalence relations on $\{0, \ldots, n-1\}$ with exactly m classes given by $f^{-1}(\lambda_0), \ldots, f^{-1}(\lambda_{m-1})$. Thus $\bigcup_{m=0}^{n} [\emptyset]\binom{n}{m}$ is the set of all equivalence relations on $\{0, \ldots, n-1\}$ and $\left(\bigcup_{m=0}^{n} [\emptyset]\binom{n}{m}, \leqq\right)$ yields the dual of the lattice $\Pi(n)$ of partitions of an n element set.

A = {0, 1}:

0-parameter words $f \in [\{0, 1\}]\binom{n}{0}$ are characteristic functions yielding subsets of $\{0, \ldots, n-1\}$. In general, parameter words $f \in [\{0, 1\}]\binom{n}{m}$ represent $\mathcal{P}(m)$ sublattices in the powerset lattice $\mathcal{P}(n)$.

For further interpretations of Graham-Rotschild parameter words compare, e.g. [14].

Notice that $\left(\bigcup_{m=0}^{n}[A]\binom{n}{m}, \leqq\right)$ for $|A|>1$ represents no lattice, since a minimal element $\mathbf{0}$ is missing.

A result of Weisner [18] says that for each two elements u, v with $u<\mathbf{1}$ of a finite lattice X the following identity is valid

$$\sum_{x\wedge u=v}\mu(x, \mathbf{1})=0.$$

This immediately yields the Möbius function μ_n^A for parameter words of length n over A:

$$\mu_n^{\emptyset}(\mathbf{0}, \mathbf{1})=(-1)^{n-1}\cdot(n-1)!$$

$$\mu_n^{\{0\}}(\mathbf{0}, \mathbf{1})=(-1)^{n}\cdot n!.$$

It is easy to see that every nonempty interval $[\pi, \tau]$ in the partition lattice $\Pi(n)$ is isomorphic to a direct product of partition lattices $\Pi(k)$ with $k\leqq n$. A similar result is valid for Graham—Rotschild parameter words: Let A be an arbitrary finite alphabet and let $f, g\in\bigcup_{m=0}^{n}[A]\binom{n}{m}$ be parameter words with $f\leqq g$. Then the interval $[f, g]$ is isomorphic to a direct product of Graham—Rothschild-parameter lattices for at most one element alphabets. This yields

Lemma. *Let A be a finite alphabet. Let $f, g\in\bigcup_{m=0}^{n}[A]\binom{n}{m}$ with $f\leqq g$. Then*

$$\mu_n^A(f, g)\neq 0.$$

By Theorem 2 this implies the following extremal result

Theorem 3. *Let A be a finite alphabet. Further let $\mathcal{G}\subseteqq\bigcup_{m=0}^{n}[A]\binom{n}{m}$ with $|\mathcal{G}|>\sum_{i=0}^{t}S_i^n(|A|)$ for some $t<n$. Then there exists a $(t+1)$-parameter word $f\in[A]\binom{n}{t+1}$ such that for every $h\in\sum_{i=0}^{t+1}[A]\binom{t+1}{i}$ there exists $g\in\mathcal{G}$ with $f\wedge g=f\cdot h$.* ∎

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